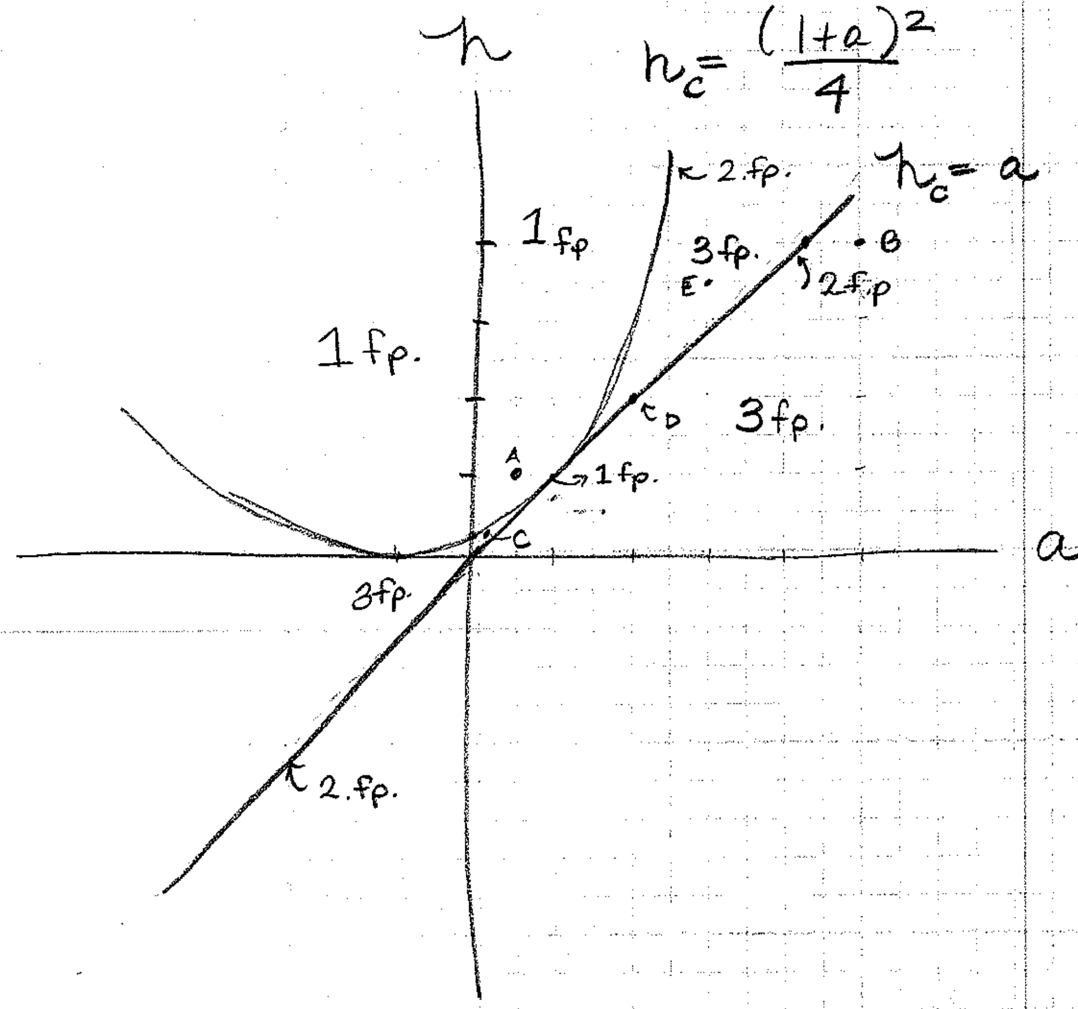


(A.) $x_1^* = 0$ is the only fixed pt.

$a = \frac{1}{2}$ $h = 1$

(B.) $a = 5$ $h = 4$ 3 Real f.p. one is $+$ stable

0

neg + Stable

(C.) $a = 0.1$ $h = 0.25$

(D) $a = 2$ $h = 2$

(E) $a = 3$ $h = 3.2$

(F) $a = 4$ $h = 4$

$x_1^* = 0 \quad x_2^* = \frac{1}{2}\left(1-a-\sqrt{1+2a+a^2-4h}\right) \quad x_3^* = \frac{1}{2}\left(1-a+\sqrt{1+2a+a^2-4h}\right)$

— one bifurcation occurs when

$$x_2^* = x_3^*$$

$$\frac{1}{2}\left(1-a-\sqrt{1+2a+a^2-4h}\right) = \frac{1}{2}\left(1-a+\sqrt{1+2a+a^2-4h}\right)$$

can only occur when

$$1 + 2a + a^2 - 4h = 0$$

$$(a+1)^2 = 4h$$

$$\boxed{h = \frac{(a+1)^2}{4}}$$

this equation gives the parameter pairs (a, h) where this bifurcation occurs

if $h > \frac{(a+1)^2}{4}$ then

$$1 + 2a + a^2 - 4h < 0$$

so x_2^* + x_3^* are imaginary

only 1 Real f.p. → $x_1^* = 0$

if $h < \frac{(a+1)^2}{4}$ then

$$1 + 2a + a^2 - 4h > 0$$

x_2^* + x_3^* are Real

∴ a saddle node occurs at $h = \frac{(1+a)^2}{4}$

the other bifurcation occurs when

$$x_1^* = x_2^* \quad \text{or} \quad x_1^* = x_3^*$$

$$0 = \frac{1}{2}\left(1 - a \pm \sqrt{1 + 2a + a^2 - 4h}\right)$$

$$1 - a = \pm\sqrt{1 + 2a + a^2 - 4h}$$

$$(1-a)^2 = 1 + 2a + a^2 - 4h$$

$$1 - 2a + \cancel{a^2} = 1 + 2a + \cancel{a^2} - 4h$$

$$1 - 4a = 1 - 4h$$

$$a = h$$

either x_2^* or x_3^* is equal to 0

when $\boxed{a = h}$ → gives the critical values of $a + h$ for the 2nd bifurcation

for $a = h$

---

something special happens at

$a = h = 1$

$$x_1^* = 0 \qquad x_2^* = \frac{1}{2}\left(1 - 1 - \sqrt{1 + 2 + 1 - 4}\right) = 0$$

$$x_3^* = \frac{1}{2}\left(1 - (1) + \sqrt{1 + 2 + 1 - 4}\right) = 0$$

all 3 fixed pts merge to become one fp at $x_1^* = x_2^* = x_3^* = 0$

$\boxed{\text{when } a = r > 1}$

e.g. $a = r = 4$

$$x_2^* = \left(\tfrac{1}{2}\left(1 - 4 - \sqrt{1 + 8 + 16 - 16}\right)\right.$$

$$= \tfrac{1}{2}\left(-3 - \sqrt{9}\right)$$

$$= \tfrac{1}{2}(-3 - 3) = -\tfrac{6}{2} = -3 \rightarrow \text{real} \neq 0$$

$$x_3^* = \tfrac{1}{2}\left(-3 + \sqrt{9}\right) = \tfrac{1}{2}(0)$$

$$= 0 = x_1^*$$

when $a = r < 1$

e.g. $a = r = 0$

$$x_2^* = \tfrac{1}{2}\left(1 - 0 - \sqrt{1 + 0 + 0 - 0}\right) =$$

$$= \tfrac{1}{2}(1 - 1) = 0 = x_1^*$$

$$x_3^* = \tfrac{1}{2}\left(1 - 0 + \sqrt{1}\right)$$

$$= \tfrac{1}{2}(1 + 1) = 1 \rightarrow \text{Real + non zero}$$

Recall what a + h are

$\boxed{h = \frac{H}{Kr}}$ → dimensionless ratio of drug killing rate to tumor growth rate normalized to the carrying capacity of the tumor

- the greater h the better the efficacy of the treatment.

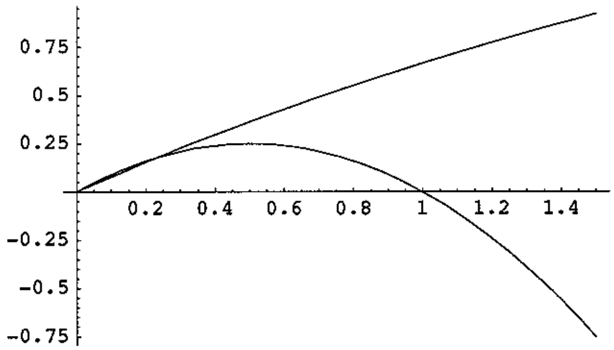
common sense:

- faster growing tumors need more potent drugs

BUT how potent is potent enough?

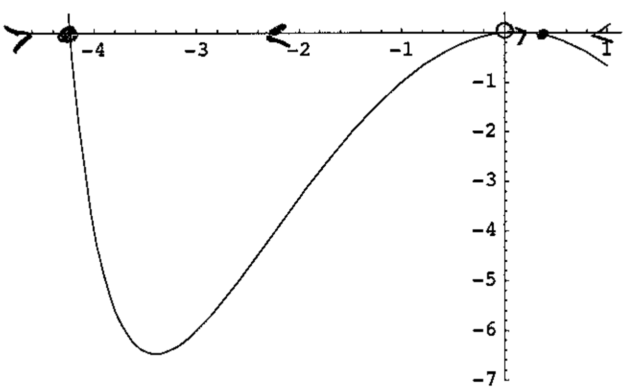
$\boxed{a = \frac{A}{K}}$ → A size of the tumor where drug is at 1/2 max efficiency relative to the carrying capacity

```
a = 5;
h = 4;
Plot[{x - x^2, h*x / (x + a)}, {x, 0, 1.5}]
NSolve[x - x^2 == h*x / (x + a), x]
Plot[{x - x^2 - (h*x / (x + a))}, {x, -4.5, 1} , PlotRange → {-7, 0.4}]
```



- Graphics -

{{x → -4.23607}, {x → 0.236068}, {x → 0.}}



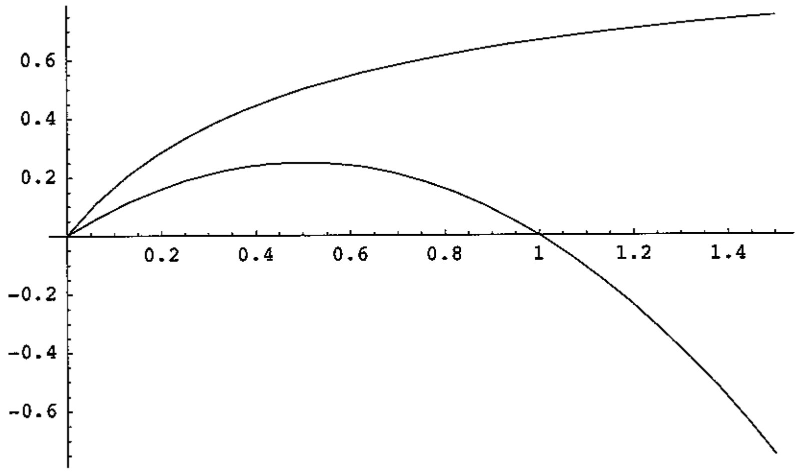
- Graphics -

Here we have 3 total real fixed points, with only 2 existing for x ≥ 0.

B.

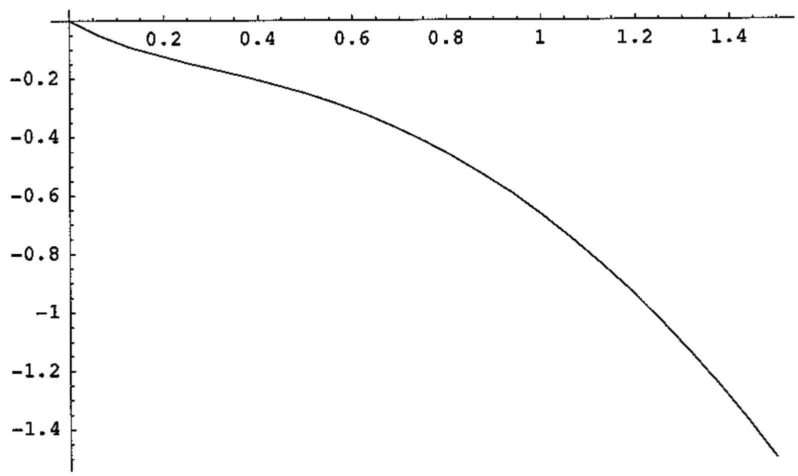
```
a = 0.5;
h = 1;
Plot[{x - x^2, h * x / (x + a)}, {x, 0, 1.5}]
NSolve[x - x^2 == h * x / (x + a), x]
Plot[{x - x^2 - (h * x / (x + a))}, {x, 0, 1.5}]
```

A



- Graphics -

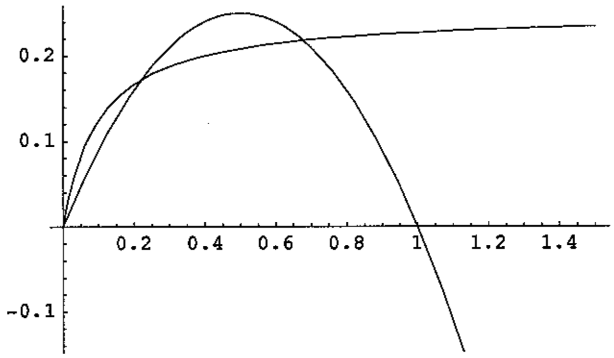
{{x → 0.25 + 0.661438 i}, {x → 0.25 - 0.661438 i}, {x → 0.}}



- Graphics -

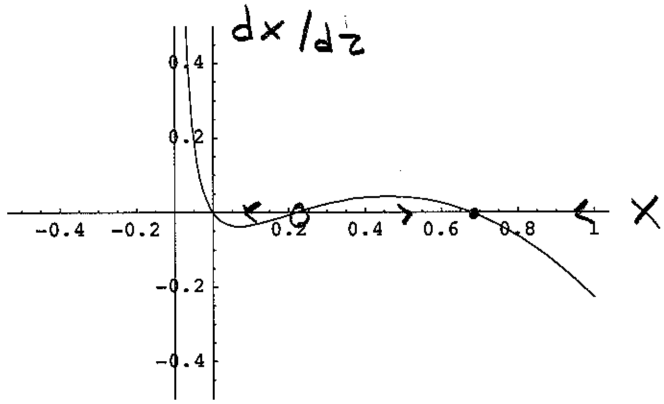
We have only 1 Real fixed point, x = 0.

```
a = 0.1;
h = 0.25;
Plot[{x - x^2, h*x / (x + a)}, {x, 0, 1.5}]
NSolve[x - x^2 == h*x / (x + a), x]
Plot[{x - x^2 - (h*x / (x + a))}, {x, -0.5, 1} , PlotRange→{-0.5, 0.5}]
```



- Graphics -

{{x → 0.679129}, {x → 0.220871}, {x → 0.}}



- Graphics -

C.

Here you have 3 fixed points all of which are ≥ 0

depends on how big the tumor is when we apply the drug!